1 Ship's Coordinate System

Axes 'forward', 'starboard' (or 'right of forward'), and 'downward' constitute, in that order, an orthogonal right-handed coordinate system. It is called the ship's coordinate system. All of the below also applies to the Lidar's coordinate system. Unit vectors aligned along the axes in the positive sense are denoted by

forward	$\widehat{x}$
starboard	$\widehat{y}$
downward	$\widehat{z}$

The angle about direction 'forward' is 'roll' ϕ, about direction 'starboard' is 'pitch' θ, about direction 'downward' is 'heading' ψ. The angular rates are $\frac{d\phi}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$. An angular rate vector can be formed by the ordered triple

$$\overrightarrow{\Omega} = \begin{pmatrix} \widehat{x} \cdot \overrightarrow{\Omega} \\ \widehat{y} \cdot \overrightarrow{\Omega} \\ \widehat{z} \cdot \overrightarrow{\Omega} \end{pmatrix} = \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} = \begin{pmatrix} \frac{d\phi}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\psi}{dt} \end{pmatrix} \tag{1}$$

which has units of radians per second. Given a position vector $\overrightarrow{r}(AB)$ from point A on the ship to point B on the ship, the cross product $\overrightarrow{\Omega} \times \overrightarrow{r}(AB)$ is

$$\overrightarrow{\Omega} \times \overrightarrow{r} = \begin{pmatrix} \widehat{x} \cdot \overrightarrow{\Omega} \times \overrightarrow{r} \\ \widehat{y} \cdot \overrightarrow{\Omega} \times \overrightarrow{r} \\ \widehat{z} \cdot \overrightarrow{\Omega} \times \overrightarrow{r} \end{pmatrix} = \begin{pmatrix} \Omega_y r_z - \Omega_z r_y \\ \Omega_z r_x - \Omega_x r_z \\ \Omega_x r_y - \Omega_y r_x \end{pmatrix} \tag{2}$$

where the components of $\overrightarrow{r}$ are in the ship coordinate system. Note that the argument (AB) of the vector $\overrightarrow{r}(AB)$ is deleted in (2). The cross product (2) is always performed in the ship's coordinate system because $\overrightarrow{r}(AB)$ is a constant in that coordinate system and $\overrightarrow{\Omega}$ is measured in that coordinate system. Alternatively, all of the above also applies to the Lidar's coordinate system.

1.1 How to Determine $\overrightarrow{\Omega} \times \overrightarrow{r}$ in the Earth's Coordinate System

Determine the correct Euler transformation matrix for instruction sheet. First, rotate about the heading axis by ψ. Second rotate about the new pitch axis by θ. Last, rotate about the new roll axis by ϕ.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \quad \text{last, rotate about the new roll axis by } \phi$$

$$B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad \text{second rotate about the new pitch axis by } \theta$$

$$C=\begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\ \text{first, rotate about the heading axis by } \psi$$

$$Q\equiv ABC=\begin{bmatrix}\cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta\\ -\cos\phi\sin\psi+\sin\theta\cos\psi\sin\phi & \cos\phi\cos\psi+\sin\theta\sin\phi\sin\psi & \cos\theta\sin\phi\\ \sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi & -\cos\psi\sin\phi+\cos\phi\sin\theta\sin\psi & \cos\theta\cos\phi\end{bmatrix}$$

The inverse of this matrix is its transpose

$$Q^{-1}=Q^{T}=\begin{bmatrix}\cos\theta\cos\psi & -\cos\phi\sin\psi+\sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi\\ \cos\theta\sin\psi & \cos\phi\cos\psi+\sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi+\cos\phi\sin\theta\sin\psi\\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi\end{bmatrix}$$

At each time step, new values of the angles ϕ, θ, ψ are determined by integration of the angular rates. At each time step the above matrix Q^{-1} is computed from the ϕ, θ, ψ. The definition of the angles gives the following transformation between components of a vector in the ship's coordinate system to the Earth's coordinate system:

$\overrightarrow{\Omega}$ as defined above is in the ship's coordinate system. Emphasize that fact by labeling it with a superscript S, that is,

$$\overrightarrow{\Omega}^{S}\equiv\overrightarrow{\Omega}$$

For example, let $\overrightarrow{\Omega}^{E}$ denote the angular rate vector when its components are in the Earth's coordinate system; then,

$$\begin{aligned}\overrightarrow{\Omega}^{S} &= Q\overrightarrow{\Omega}^{E}\\ \overrightarrow{\Omega}^{E} &= Q^{-1}\overrightarrow{\Omega}^{S}\end{aligned}$$

The required computation at each time step is to calculate the components of $\overrightarrow{\Omega}\times\overrightarrow{r}$ in the Earth's coordinate system which is denoted by $\left(\overrightarrow{\Omega}\times\overrightarrow{r}\right)^{E}$. Since (2) is $\left(\overrightarrow{\Omega}\times\overrightarrow{r}\right)^{S}$, we obtain $\left(\overrightarrow{\Omega}\times\overrightarrow{r}\right)^{E}$ from

$$\left(\overrightarrow{\Omega}\times\overrightarrow{r}\right)^{E}=Q^{-1}\left(\overrightarrow{\Omega}\times\overrightarrow{r}\right)^{S} \tag{1}$$

1.2 How to Determine the Radar's Radial Direction in the Earth's Coordinate System and Calculate the Antenna's Radial Velocity

First, define the radar radial unit vector in the ship's coordinate system. Assume that the radar's measurement of azimuth φ is level with the main deck, and zero degrees azimuth is forward, and azimuth is positive if the rotation is from forward toward starboard. Assume that the, radar's measurement of elevation ε is positive from the plane containing the main deck. Then the unit vector pointing outward from the radar's antenna in the ship's coordinate system is

$$\widehat{p}^{S}=\begin{pmatrix}\widehat{x}\cdot\widehat{p}\\ \widehat{y}\cdot\widehat{p}\\ \widehat{z}\cdot\widehat{p}\end{pmatrix}=\begin{pmatrix}p_x\\ p_y\\ p_z\end{pmatrix}=\begin{pmatrix}\cos\varphi\cos\varepsilon\\ \sin\varphi\cos\varepsilon\\ -\sin\varepsilon\end{pmatrix} \tag{2}$$

Verify that it is a unit vector:

$$\widehat{p}^S \cdot \widehat{p}^S = \left(\cos^2\varphi + \sin^2\varphi\right)\cos^2\varepsilon + \sin^2\varepsilon = \cos^2\varepsilon + \sin^2\varepsilon = 1$$

Similar to (1), the radar radial unit vector in the earth's coordinate system is

$$\widehat{p}^E = Q^{-1}\widehat{p}^S \tag{3}$$

Assume the convention that motion toward the radar antenna is negative and motion away from the radar antenna is positive. The radial velocity correction in Earth coordinates is

$$\widehat{p}^E \cdot \overrightarrow{v}_S^E$$

This correction must be added to (not subtracted from) the radar's measurement of radial velocity.

Assume that the Univ. Miami radars are pointed straight up relative to the main deck. Then

$$\widehat{p}^S = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

The ship's heave is the velocity component is perpendicular to the main deck; heave is positive for downward motion. The correction is the negative of the ship's local heave at the location of the radar antenna, namely

$$\widehat{p}^E \cdot \overrightarrow{v}_S^E = \left(Q^{-1}\widehat{p}^S\right) \cdot \overrightarrow{v}_S^E =$$

$$\begin{aligned}
&(-\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi)\left(v_S^E\right)_x \\
&+ (\cos\psi\sin\phi - \cos\phi\sin\theta\sin\psi)\left(v_S^E\right)_y \\
&+ (-\cos\theta\cos\phi)\left(v_S^E\right)_z
\end{aligned}$$

Because

$$Q^{-1}\widehat{p}^S =$$

$$\begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi \\ \cos\psi\sin\phi - \cos\phi\sin\theta\sin\psi \\ -\cos\theta\cos\phi \end{pmatrix}$$

More generally, for (2) we have that $\widehat{p}^E = Q^{-1}\widehat{p}^S$ is

$$\begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\varepsilon \\ \sin\varphi\cos\varepsilon \\ -\sin\varepsilon \end{pmatrix} =$$

$$\begin{pmatrix} \cos\theta\cos\varepsilon\cos\psi\cos\varphi - (\sin\varepsilon)(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi) + (\cos\varepsilon\sin\varphi)(-\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi) \\ \cos\theta\cos\varepsilon\cos\varphi\sin\psi + (\cos\varepsilon\sin\varphi)(\cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi) - (\sin\varepsilon)(-\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi) \\ -\cos\theta\cos\phi\sin\varepsilon - \sin\theta\cos\varepsilon\cos\varphi + \cos\theta\cos\varepsilon\sin\phi\sin\varphi \end{pmatrix}$$

1.3 How to Calculate the Spatial Position of each Datum of Radar or Lidar Measurement

Radius of the Earth is $R_{Earth} = 6378$ km. Note that the GPS geoid is about 35 m above sea level in the RICO study area; that corresponds to altitude $z = -35$ m in ship's coordinates. The NOAA/K radar and the LIDAR measure each datum within their averaging volume at their recorded range r_{ange}, elevation ε, and azimuth φ. The Univ. Miami radar records only range because its elevation is fixed at vertical. The position vector of each datum relative to the antenna in the Earth's coordinate system is $\widehat{p}^E r_{ange}$. See (3) above for the calculation of $\widehat{p}^E$. The ship's POS MV system gives latitude, longitude and altitude of the center of the radars' antennas and of the lidar's IMU as functions of time: $(lat\,(t)\,, lon\,(t)\,, z\,(t))$. The unit of $lat\,(t)$ and $lon\,(t)$ is decimal degrees, and the unit of $z\,(t)$ is meters. Increments of latitude and longitude are calculated in the small angle approximation, e.g., $\sin\left(r_{ange}/R_{Earth}\right) \simeq r_{ange}/R_{Earth}$. The increment of latitude in degrees associated with the north component of $\widehat{p}^E r_{ange}$ is

$$\Delta lat = \frac{180}{\pi R_{Earth}} \left(\widehat{x} \cdot \widehat{p}^E r_{ange}\right) = \frac{180}{\pi R_{Earth}} \left(\widehat{p}^E\right)_x r_{ange}$$

The increment of longitude associated with the east component of $\widehat{p}^E r_{ange}$ is

$$\Delta lon = \frac{180}{\pi R_{Earth} \cos\left(lat\right)} \left(\widehat{y} \cdot \widehat{p}^E r_{ange}\right) = \frac{180}{\pi R_{Earth} \cos\left(lat\right)} \left(\widehat{p}^E\right)_y r_{ange}$$

The increment of altitude is

$$\begin{aligned} \Delta z &= \left(\widehat{z} \cdot \widehat{p}^E\right) r_{ange} \\ &= \left(\widehat{p}^E\right)_z r_{ange} \end{aligned}$$

Recall that $\widehat{z}$ points downward.

Finally, the latitude, longitude, and altitude of each datum of radars and lidar is

$$\begin{aligned} lat_{datum} &= lat\,(t) + \Delta lat \\ lon_{datum} &= lon\,(t) + \Delta lon \\ z_{datum} &= z\,(t) + \Delta z \end{aligned}$$

The height H above the sea surface for each datum can be obtained to within about 0.2 m given the fact that the main deck is 1.2 m above the sea surface. Using the surveyed heights of the radars and lidar given in "Coordinates for IMU," we have

$$\begin{aligned} H_{lidar} &= 1.2 + 4.84 - (\Delta z)_{lidar} \\ H_{NOAA/K} &= 1.2 + 5.30 - (\Delta z)_{NOAA/K} \\ H_{W-band} &= 1.2 + 2.88 - (\Delta z)_{W-band} \end{aligned}$$

These heights are given in meters and are positive for positions above the sea surface; Δz is subtracted above because Δz is negative for positions above the sea surface.